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HYDRODYNAMIC RESISTANCE OF A SPHEROIDAL PARTICLE WITH UNIFORM INTERNAL HEAT RELEASE

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The Stokes approximation is used to describe the stationary motion of a heated hydrosol spheroidal particle in a viscous incompressible liquid in which internal, uniformly distributed heat sources (sinks) of constant capacity act. It was assumed that the average particle surface temperature could differ significantly from the temperature of the ambient liquid. An analytical expression for the hydrodynamic force acting on the uniformly heated spheroidal particle was obtained by solving hydrodynamic equations with the temperature dependence of the viscosity represented as an exponential power series.

1. Formulation of the Problem. The motion of heated particles in viscous liquids and gases was considered in [1–5]. By a heated particle we understand a particle whose average surface temperature far exceeds the ambient temperature. Heating of the particle surface can be caused by a bulk chemical reaction, radioactive decay of the particle, etc.

The heated surface of the spheroid can significantly affect the thermal-physics properties of the ambient medium and, therefore, the velocity and pressure distributions in the vicinity of the particle.

At present the motion of rigid spheroidal particles under a small relative temperature gradient in their vicinity has been studied in considerable detail [6–8].

In the present paper, an analytical expression for the hydrodynamic force acting on a heated spheroidal particle is obtained in the Stokes approximation, allowing for the temperature dependence of viscosity represented as an exponential power series under arbitrary temperature gradients between the particle surface and distant areas.

We consider the motion of a hydrosol rigid particle shaped as an oblate spheroid in a viscous incompressible liquid in which constant-capacity heat sources (sinks) are uniformly distributed. The Reynolds numbers are small. The particle moves along the axis of symmetry under the action of a certain force, say, an electromagnetic force. If we convert to a coordinate system attached to the particle, the problem reduces to that of a plane-parallel liquid flow with velocity $\boldsymbol{U}_\infty$ ($\boldsymbol{U}_\infty \parallel Oz$) past a heated motionless oblate (prolate) spheroid.

The density, thermal conductivity, and heat capacity of the liquid and the particle are assumed to be constant, and the thermal conductivity exceeds the thermal conductivity of the ambient liquid.

Among all liquid transport parameters, only dynamic viscosity depends strongly on temperature [9]. The temperature dependence of the viscosity is written as

$$\mu_{\rm liq} = \mu_\infty \Big[1 + \sum_{n=1}^{\infty} F_n \Big(\frac{T_{\rm liq}}{T_\infty} - 1\Big)^n\Big] \exp\Big(-A\Big(\frac{T_{\rm liq}}{T_\infty} - 1\Big)\Big), \tag{1.1}$$

where $A = \text{const}$, $\mu_\infty = \mu_{\rm liq}(T_\infty)$, and T_∞ is the liquid temperature at a distance from the particle. Hereinafter the subscripts "liq" and "p" stand for the ambient liquid and the particle. For $F_n = 0$, formula (1.1) can be reduced to the Reynolds relation (ratio) [9].

As is known, liquid viscosity decreases exponentially as temperature rises [9]. An analysis of semi-empirical formulas available in the literature shows that expression (1.1) provides for the most adequate description of viscosity variation over a wide temperature range. Thus, for water over the temperature range of 0–90°C, the parameter

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values in Eq. (1.1) are as follows: $A = 5.779$, $F_1 = -2.318$, and $F_2 = 9.118$ at $T_\infty = 273$ K (the relative error does not exceed 3%).

The flow past the spheroid is described in a spherical coordinate system $(\varepsilon, \eta, \varphi)$ whose origin is at the center of the hydrosol particle. The curvilinear coordinates ε, η, and φ are linked to the Cartesian coordinates by the following relations [10]:

— for the case of a prolate spheroid ($a_0 < b_0$),

$$x = c \sinh \varepsilon \, \sin \eta \, \cos \varphi, \quad y = c \sinh \varepsilon \, \sin \eta \, \sin \varphi, \quad z = c \cosh \varepsilon \, \cos \eta, \quad c = \sqrt{b_0^2 - a_0^2}; \tag{1.2}$$

— for the case of an oblate spheroid ($a_0 > b_0$),

$$x = c \cosh \varepsilon \, \sin \eta \, \cos \varphi, \quad y = c \cosh \varepsilon \, \sin \eta \, \sin \varphi, \quad z = c \sinh \varepsilon \, \cos \eta, \quad c = \sqrt{a_0^2 - b_0^2}. \tag{1.3}$$

Here a_0 and b_0 are the semiaxes of the spheroid. In the Cartesian system of coordinates, the z axis coincides with the axis of symmetry of the spheroid.

For small Reynolds and Peclet numbers, the distributions of the velocity $\boldsymbol{U}_{\rm liq}$, pressure $P_{\rm liq}$, and temperature $T_{\rm liq}$ are described by the system of equations for lower gravity [10]:

$$\nabla P_{\rm liq} = \mu_{\rm liq} \Delta \boldsymbol{U}_{\rm liq} + 2(\nabla \mu_{\rm liq} \nabla) \boldsymbol{U}_{\rm liq} + [\nabla \mu_{\rm liq} \times \operatorname{rot} \boldsymbol{U}_{\rm liq}], \qquad \operatorname{div} \boldsymbol{U}_{\rm liq} = 0, \tag{1.4}$$

$$\Delta T_{\rm liq} = 0, \qquad \Delta T_p = -q_p/\lambda_p. \tag{1.5}$$

The boundary conditions for system (1.4), (1.5) have the following form:

$$\varepsilon = \varepsilon_0, \qquad \boldsymbol{U}_{\rm liq} = 0, \quad T_{\rm liq} = T_p, \quad \lambda_{\rm liq} \frac{\partial T_{\rm liq}}{\partial \varepsilon} = \lambda_p \frac{\partial T_p}{\partial \varepsilon}, \tag{1.6}$$

$$\varepsilon \to \infty, \qquad \boldsymbol{U}_{\rm liq} \to U_\infty \boldsymbol{e}_\varepsilon \cos \eta - U_\infty \boldsymbol{e}_\eta \sin \eta, \quad T_{\rm liq} \to T_\infty, \quad P_{\rm liq} \to P_\infty, \tag{1.7}$$

$$\varepsilon \to 0, \qquad T_p \neq \infty. \tag{1.8}$$

Here q_p is the constant capacity of the heat sources (sinks) per unit volume of the particle, $\boldsymbol{e}_\varepsilon$ and $\boldsymbol{e}_\eta$ are unit vectors of the spherical coordinate system, λ is the thermal conductivity, and $U_\infty = |\boldsymbol{U}_\infty|$.

Boundary conditions (1.6) correspond to the slip condition for the velocity, temperature equality, and heat flux continuity on the particle surface. The coordinate surface with the value of $\varepsilon = \varepsilon_0$ corresponds to the particle surface. At a large distance from the particle ($\varepsilon \to \infty$), boundary conditions (1.7) hold, and the finiteness of the physical quantities characterizing the particle at $\varepsilon \to 0$ is allowed for in (1.8).

The force acting on the particle from the flow is given by the formula

$$F_z = \int\limits_S \Big(- P_{\rm liq} \cos \eta + \sigma_{\varepsilon\varepsilon} \cos \eta - \frac{\sinh \varepsilon}{\cosh \varepsilon} \, \sigma_{\varepsilon\eta} \sin \eta \Big) \, dS, \tag{1.9}$$

where $dS = c^2 \cosh^2 \varepsilon \, \sin \eta \, d\eta \, d\varphi$ is a differential element of the surface and $\sigma_{\varepsilon\varepsilon}$ and $\sigma_{\varepsilon\eta}$ are stress tensor components in spherical coordinates [11].

Using boundary condition (1.7), we seek expressions of the normal (U_ε) and tangential (U_η) components of the mass velocity $\boldsymbol{U}_{\rm liq}$ in the form

$$U_\varepsilon(\varepsilon, \eta) = \frac{U_\infty}{c H_\varepsilon \cosh \varepsilon} \, G(\varepsilon) \cos \eta, \qquad U_\eta(\varepsilon, \eta) = -\frac{U_\infty}{c H_\varepsilon} \, g(\varepsilon) \sin \eta, \tag{1.10}$$

where $G(\varepsilon)$ and $g(\varepsilon)$ are arbitrary functions of the normal coordinate ε and H_ε is the Lamés coefficient in spherical coordinates [10].

2. Velocity Field and Temperature Distribution. Obtaining the Resistance Force. To find the force acting from the liquid on the solid heated spheroidal particle, we need to know the distributions of temperature, mass velocity, and pressure in the vicinity of the particle. Integrating Eqs. (1.5) with the corresponding boundary conditions, we obtain

$$t_{\rm liq} = 1 + (\gamma/c) \operatorname{arccot} \lambda, \tag{2.1}$$

$$t_p = B + \frac{\lambda_{\rm liq}}{\lambda_p} \frac{\gamma}{c} \operatorname{arccot} \lambda + \int\limits_{\lambda_0}^{\lambda} \frac{\operatorname{arccot} \lambda}{c} \, f \, d\lambda - \frac{\operatorname{arccot} \lambda}{c} \int\limits_{\lambda_0}^{\lambda} f \, d\lambda.$$

Here $\lambda = \sinh\varepsilon$, $t = T/T_\infty$, $\gamma = t_{\rm s} - 1$ is a dimensionless parameter that characterizes the particle surface heating, $t_{\rm s} = T_{\rm s}/T_\infty$, $T_{\rm s}$ is the average surface temperature of the heated spheroid defined by the formula $T_{\rm s}/T_\infty = 1 + a_0 b_0 q_p/(3\lambda_{\rm liq} T_\infty)$, $B = 1 + (1 - \lambda_{\rm liq}/\lambda_p)\gamma\sqrt{1+\lambda_0^2}\,\mathrm{arccot}\,\lambda_0$, $\lambda_0 = \sinh\varepsilon_0$, $f = -\dfrac{c^2}{2\lambda_p T_\infty}\displaystyle\int_{-1}^{+1} q_p(\lambda^2 + x^2)\,dx$, and $x = \cos\eta$.

By virtue of (2.1), formula (1.1) becomes

$$\mu_{\rm liq} = \mu_\infty\Big[1 + \sum_{n=1}^{\infty} F_n\Big(\frac{\gamma}{c}\ \mathrm{arccot}\ \lambda\Big)^n\Big]\exp\left(-\gamma_0\,\mathrm{arccot}\ \lambda\right) \qquad \Big(\gamma_0 = \frac{A\gamma}{c}\Big).$$

Because the viscosity depends only on the radial coordinate λ, the system of hydrodynamic equations (1.4) is solved by the method of separation of variables taking into account (1.10). Specifically, for the mass velocity components $\boldsymbol{U}_{\rm liq}$, we obtained the following expressions subject to the boundary conditions at infinity (1.7):

$$U_\varepsilon(\varepsilon,\eta) = \frac{U_\infty}{cH_\varepsilon}[c^2 + A_1G_1 + A_2G_2]\cos\eta, \quad U_\eta(\varepsilon,\eta) = -\frac{U_\infty}{cH_\varepsilon}[c^2 + A_1G_3 + A_2G_4]\sin\eta, \tag{2.2}$$

where

$$G_1 = -\frac{1}{\lambda^3}\sum_{n=0}^{\infty}\frac{\theta_n^{(1)}}{(n+3)\,\lambda^n}, \quad G_2 = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{\theta_n^{(2)}}{(n+1)\lambda^n} - \frac{\beta}{\lambda^3}\sum_{n=0}^{\infty}\frac{\theta_n^{(1)}}{(n+3)\lambda^n}\Big[(n+3)\ln\frac{\lambda_0}{\lambda} - 1\Big],$$

$$G_3 = G_1 + \frac{1+\lambda^2}{2\lambda}G_1^{\rm I}, \qquad G_4 = G_2 + \frac{1+\lambda^2}{2\lambda}G_2^{\rm I},$$

$$\theta_n^{(1)} = -\frac{1}{n(n+5)}\sum_{k=1}^{n}[(n+4-k)\{(n+1-k)\alpha_k^{(1)} + \alpha_k^{(2)}\} + \alpha_k^{(3)}]\theta_{n-k}^{(1)} \qquad (n \geqslant 1),$$

$$\theta_n^{(2)} = -\frac{1}{(n-2)(n+3)}\Big[\sum_{k=1}^{n}\{(n+2-k)[(n+1-k)\alpha_k^{(1)} + \alpha_k^{(2)}] + \alpha_k^{(3)}\}\theta_{n-k}^{(2)}$$

$$+ \beta\sum_{k=0}^{n}[(2n-2k+3)\alpha_k^{(1)} + \alpha_k^{(1)}]\theta_{n-k-2}^{(1)} - 6\alpha_n^{(4)}\Big] \qquad (n \geqslant 3),$$

$$H_\varepsilon = c\sqrt{\cosh^2\varepsilon - \sin^2\eta},$$

$$\theta_1^{(2)} = -[2(\alpha_1^{(1)} + \alpha_1^{(2)}) + \alpha_1^{(3)} + 6\alpha_1^{(4)}]/4, \quad \theta_2^{(2)} = 1, \quad \theta_0^{(1)} = -1, \quad \theta_0^{(2)} = -1,$$

$$\beta = -[\{3(2\alpha_1^{(1)} + \alpha_1^{(2)}) + \alpha_1^{(3)}\}\theta_1^{(2)} - 2(\alpha_2^{(1)} + \alpha_2^{(2)}) - \alpha_2^{(3)} - 6\alpha_2^{(4)}]/5,$$

$$\alpha_n^{(1)} = C_n + 12\sum_{k=0}^{n_2}(-1)^k\frac{C_{n-2k-2}}{(2k+1)(2k+3)(2k+5)}, \qquad \alpha_n^{(4)} = \Delta_n, \quad \Delta_0 = 1, \quad \Delta_1 = \gamma,$$

$$\alpha_n^{(2)} = (n-2)C_n - \gamma_0 C_{n-1} + 12\sum_{k=0}^{n_2}(-1)^k\frac{(4k+5)C_{n-2k-2}}{(2k+1)(2k+3)(2k+5)}$$

$$-\,3\sum_{k=0}^{n_3}(-1)^k\frac{1}{(2k+3)(2k+5)}[(n-2k-2)C_{n-2k-2} - \gamma_0 C_{n-2k-3} + (n-2k-4)C_{n-2k-4}] \quad (n \geqslant 1),$$

$$\alpha_n^{(3)} = -2(n+2)C_n + 2\gamma_0 C_{n-1} - 2(n-2)C_{n-2} + 12\sum_{k=0}^{n_2}(-1)^k\frac{C_{n-2k-2}}{2k+5}$$

$$+\,6\sum_{k=0}^{n_3}(-1)^k\frac{(k+2)(4k+5)}{(2k+3)(2k+5)}[(n-2k-2)C_{n-2k-2} - \gamma_0 C_{n-2k-3} + (n-2k-4)C_{n-2k-4}] \quad (n \geqslant 1),$$

TABLE 1

a_0/b_0	K at temperature T_s						
	273 K	283 K	303 K	333 K	343 K	353 K	363 K
0.73	0.947	0.705	0.393	0.163	0.121	0.089	0 .065
0.90	0.980	0.727	0.397	0.158	0.116	0.086	0 .062

TABLE 2

a_0/b_0	K	
	$T_s = 283$ K	$T_s = 333$ K
0.71	0.5822	0.1451
0.75	0.7076	0.1614
0.80	0.7137	0.1594
0.85	0.7201	0.1585
0.90	0.7266	0.1581
0.95	0.7332	0.1582
0.99	0.7386	0.1585

$$\Delta_{n+2} = \frac{1}{n+2}[\gamma_0 \Delta_{n+1} - n\Delta_n] \qquad (n \geqslant 0),$$

$$C_k = \sum_{l_1+3l_3+5l_5+\ldots+sl_s=k} \frac{l!}{l_1!l_3!l_5!\cdots l_s!} F_{\rm liq} f_1^{l_1} f_3^{l_3} f_5^{l_5} \cdots f_s^{l_s}, \quad C_0 = 1, \quad s = k - \frac{1+(-1)^k}{2},$$

$$l = l_1 + l_3 + l_5 + \ldots + l_s, \quad f_{2k-1} = (-1)^{k-1}\frac{\gamma}{c(2k-1)} \quad (k \geqslant 1), \quad n_x = \Big[\frac{n+x}{2}\Big].$$

In particular, $C_1 = F_1\gamma/c$, $C_2 = F_2\gamma^2/c^2$, $C_3 = F_3\gamma^3/c^3 - F_1\gamma/(3c)$, and $C_4 = F_4\gamma^4/c^4 - 2F_2\gamma^2/(3c^2)$. The integer part of the number $k/2$ is denoted by $[k/2]$.

The integration constants A_1 and A_2 are obtained from the boundary conditions on the spheroid surface

$$A_1 = -c^2 \frac{G_2^{\rm I}}{G_1G_2^{\rm I} - G_2G_1^{\rm I}}, \qquad A_2 = c^2 \frac{G_1^{\rm I}}{G_1G_2^{\rm I} - G_2G_1^{\rm I}}, \tag{2.3}$$

where $G_1^{\rm I} = dG_1/d\lambda$ and $G_2^{\rm I} = dG_2/d\lambda$ are the first derivatives of the corresponding functions with respect to λ.

Integrating expressions (1.9) over the spheroid surface and taking into account (2.2), we obtain the force acting on the spheroid due to viscous stress:

$$\boldsymbol{F}_z = -4\pi \frac{\mu_\infty U_\infty}{c} A_2 \exp\Big(-\frac{A\gamma}{c} \operatorname{arccot} \lambda_0\Big)\boldsymbol{n}_z, \tag{2.4}$$

where $\boldsymbol{n}_z$ is a unit vector along the z axis.

Expression (2.4) is obtained under the assumption of uniform particle motion, which is possible only when the total force acting on the particle is zero. Since the force (2.4) is proportional to the velocity and becomes zero together with it, uniform motion of a heated oblate spheroid occurs only in the presence of a certain external force that balances force (2.4), e.g., an electromagnetic force.

Allowing for (2.3), expression (2.4) can be written as

$$\boldsymbol{F}_z = 6\pi a_0 \mu_\infty K U_\infty \boldsymbol{n}_z, \tag{2.5}$$

where $K = [2G_1^{\rm I}/(3\sqrt{1+\lambda_0^2}[G_2G_1^{\rm I} - G_1G_2^{\rm I}])] \exp(-(A\gamma/c) \operatorname{arccot} \lambda_0)$. In the expressions for G_1, G_2, $G_1^{\rm I}$, and $G_2^{\rm I}$, $\lambda = \lambda_0$. To find the expression for the hydrodynamical resistance of an oblate spheroid, in (2.5) we must replace λ by $i\lambda$ and c by $-ic$ (i is imaginary unity).

Formula (2.5) has a general nature and describes the hydrodynamic force acting on a highly heat-conducting spheroidal particle with internal heat sources (sinks) and arbitrary temperature dependence of viscosity.

The influence of the particle form factor and its surface temperature on the resistance force is determined by the coefficient K. Tables 1and 2 give results of numerical calculation of the coefficient K with variation in the average temperature of the spheroid surface and the ratio of semiaxes for solid particles suspended in water at

$T_\infty = 273$ K ($A = 5.779$ and $F_n = 0$, where $n \geqslant 1$). An analysis of the numerical results shows that the heating of the spheroid surface significantly affects the resistance force.

For $\gamma \to 0$ (small temperature gradients in the vicinity of the spheroid), $G_1 = 1/(3\lambda^3)$, $G_1^{\mathrm{I}} = -1/\lambda^4$, $G_2 = 1/\lambda$, $G_2^{\mathrm{I}} = -1/\lambda^2$, $a_0 = b_0 = R$, and $K = 1$; formula (2.5) becomes the Stokes formula for a rigid spherical particle with radius R [11].

We now consider the motion of a uniformly heated spheroidal particle with average surface temperature T_{s}. This problem can be solved using the results obtained above. In particular, if an electromagnetic radiation flux with intensity I_0 and wavelength $\tilde{\lambda}_0$ is incident on the spheroid, the energy absorbed by the particle is $\pi R^2 I_0 K_n$ (R is the major semiaxis of the spheroid and K_n is the absorption coefficient) [12]. Let us assume that $\tilde{\lambda}_0 \gg R$. Then, the absorbed energy is uniformly distributed over the particle surface, i.e., it can be considered uniformly heated. In this case, we must set $q_p = 0$ and assume $T_{\mathrm{liq}} = T_{\mathrm{s}}$ in boundary conditions (1.6). The parameter γ, which characterizes the relative temperature gradient between the particle surface and the region away from it, has the form $\gamma = c(t_{\mathrm{s}} - 1)/\operatorname{arccot} \lambda_0$, where $t_{\mathrm{s}} = T_{\mathrm{s}}/T_\infty$.

Thus, we obtained a formula for the resistance force of a spheroidal particle with arbitrary temperature gradients in its vicinity taking into account the temperature dependence of the viscosity represented as an exponential series.

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